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Monte Carlo Estimation Under Different Distributions Using the Same Simulation

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Two methods for reducing the computer time necessary to investigate changes in distribution of random inputs to large simulation computer codes are presented. The first method produces unbiased estimators of functions of the output variable under the new distribution of the inputs. The second method generates a subset of the original outputs that has a distribution corresponding to the new distribution of inputs. Efficiencies of the two methods are examined.

KEY WORDS: Computer models; Sampling; Sensitivity analysis; Importance sampling.

1. INTRODUCTION AND SUMMARY

Long-running computer codes have been used in assessing the risks and benefits of such things as nuclear power and hazardous waste disposal. Examples of these may be found in Dillon, Lantz, and Pahwa (1978); Hirt and Romero (1975); and McKay, Conover, and Beckman (1979). An additional study is found in the example given by Iman and Conover (1980), and such studies were suggested by Goodman and Koch (1982) and Levinson and Yeater (1983). Although the codes are intrinsically deterministic, incomplete knowledge of the modeled process, often expressed as uncertainty about values of the input variables, leads one to treat the inputs as random variables. The analyst then interprets the output itself as a random variable, the distribution of which is related to that of the inputs by the transformation of the computer code.

When the analyst is faced with defending the choice of the distribution of the inputs and assessing its influence on conclusions, costs may make rerunning the code with new distributions prohibitive. Nevertheless, the sensitivity of estimates of the mean, cdf, or other statistics to changing the distribution of the inputs will need to be known. The purpose of this article is to give two methods that allow the analyst to change the distribution of the input variables without rerunning the computer code.

Mathematically the problem can be described as follows. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be independent (sample) vectors of input variables with density $f_1(\mathbf{x})$, and let $Y_i = H(\mathbf{X}_i)$ be the output variable for input vector $\mathbf{X}_i$. Let H represent the computer code or any transformation of the output from the computer code. For clarity of presentation, assume that the expected value of Y is the parameter to be estimated. This expectation is denoted by $\mu_1 = E_{f_1}\{Y\}$ when the inputs come from f_1 and by $\mu_2 = E_{f_2}\{Y\}$ when they have a different density, $f_2(\mathbf{x})$. Two methods are given for changing the distribution of the inputs after they are sampled from f_1. The first method is a weighting scheme similar to importance sampling (see Kahn and Marshall 1953), and it produces an unbiased estimator of μ_2. The other method is a rejection method (see Kennedy and Gentle 1980) that leaves the output variable $H(\mathbf{X})$ with the density induced by letting $\mathbf{X}$ have density f_2.

Two methods and their efficiencies, relative to direct simulation, are examined in Sections 2 and 3. The efficiencies of each method are shown to decrease rapidly as large differences occur between the densities $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$. The efficiency of the weighting method, however, may be greater than 1.0 for smaller differences between the two densities.

Although there is usually a loss of efficiency associated with the application of these methods, they do offer the analyst a way to investigate the sensitivity of the system under study (the computer code) to the simulation distribution. This sensitivity study can be made at almost no cost compared to that of rerunning the computer code for all desired choices of f_2.

An example of a simulation of the Three Mile Island Unit 2 reactor is given in Section 4. Additional examples are given in Sections 2 and 3 to illustrate the application and properties of the two methods.

2. THE WEIGHTING METHOD

We assume that the support R_1 for the density function f_1 contains the support R_2 for the density f_2. It is also assumed that $H(\mathbf{X}_i)$ $(i = 1, \ldots, n)$ have been simulated using f_1 and that an estimate of $\mu_2 =$

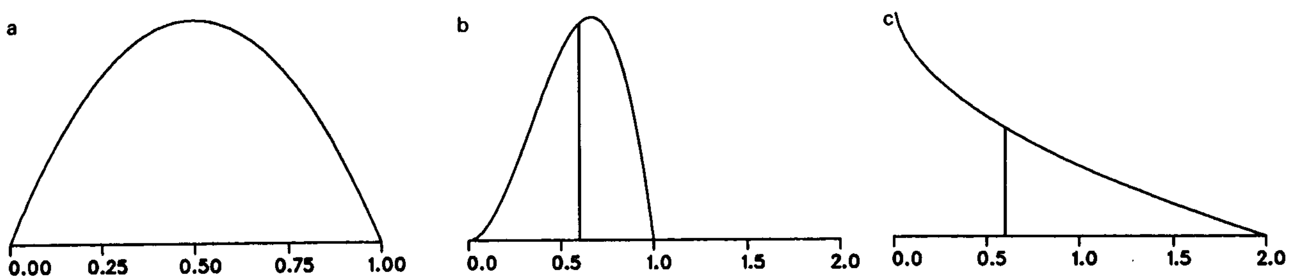

Figure 1. Relevant Densities for the Weighting Method: (a) Density of H (X) Under f_1; (b) Density of H (X) Under f_2; (c) Density of w (X) H (X) Under f_1.

$E_{f_2}\{H(\mathbf{X})\}$ is desired. Let $\mathbf{Y}$ be the vector of the n outputs from the simulation, $Y_i = H(\mathbf{X}_i)$, and let $w_i = f_2(\mathbf{x}_i)/f_1(\mathbf{x}_i)$. Our estimator of μ_2 is $g(\mathbf{Y}) = \Sigma w_i Y_i/n$, a weighted average of the original outputs. To see that $g(\mathbf{Y})$ is an unbiased estimator of μ_2, observe that

$$E_{f_1}(g(\mathbf{Y})) = \left\{\int_{R_1} \Sigma w_i H(\mathbf{x}_i) f_1(\mathbf{x}_i)\, d\mathbf{x}_i\right\}\Big/ n$$
$$= \left\{\Sigma \int_{R_1} H(\mathbf{x}_i) f_2(\mathbf{x}_i) f_1(\mathbf{x}_i)/f_1(\mathbf{x}_i)\, d\mathbf{x}_i\right\}\Big/ n.$$

Since $R_2 \subset R_1$ and f_2 is 0 outside of R_2, we have

$$E_{f_1}(g(\mathbf{Y})) = \left\{\Sigma \int_{R_2} H(\mathbf{x}_i) f_2(\mathbf{x}_i)\, d\mathbf{x}_i\right\}\Big/ n$$
$$= E_{f_2}\{H(\mathbf{X})\} = \mu_2.$$

A schematic diagram representing this weighting method is given in Figure 1. Figure 1(a) shows the density of $H(\mathbf{X})$ when the underlying density is f_1. The density of $H(\mathbf{X})$ with an underlying distribution of f_2 is given in Figure 1(b) with the mean value marked with a vertical line. The density function of the transformation $w(\mathbf{X})H(\mathbf{X})$ under f_1 is given in Figure 1(c) for a linear weight function $w(\mathbf{x})$. The vertical line denotes the mean of the density. The support of this density is from 0 to 2, as the maximum value of $w(\mathbf{X})$ is 2. From Figure 1(b and c) it is obvious that the density function of $H(\mathbf{X})$ under sampling from f_2 and the density function of $w(\mathbf{X})H(\mathbf{X})$, where $\mathbf{X}$ is generated from f_1, are not the same. The mean values are the same, however; that is, $E_{f_1}\{w(\mathbf{X})H(\mathbf{X})\} = E_{f_2}\{H(\mathbf{X})\}$.

Although this method closely resembles importance sampling, its intent is different. In importance sampling, a density f_1 is used for sampling in such a way that the resulting estimator of μ_2 is unbiased and has smaller variance than that obtainable from simple random sampling from f_2. Since in this case f_1 and f_2 are fixed, smaller or larger variances could result depending on the situation.

The efficiency of $g(\mathbf{Y})$ is measured by its variance relative to the variance of $\Sigma Y_i/n$ when the corresponding values of $\mathbf{X}$ come from f_2. For simulations with large differences in the shapes of the two densities, the efficiency is usually near 0. This arises because the weight function $w(\mathbf{x})$ becomes large over wide ranges of the vector $\mathbf{x}$.

To assist in the study of the efficiency of this method, we assume that the expected values of H and H^2 with respect to densities indexed by a parameter (vector) $\boldsymbol{\theta}$ may be expressed as $E_{\boldsymbol{\theta}}\{H(\mathbf{X})\} = \phi(\boldsymbol{\theta})$ and $E_{\boldsymbol{\theta}}\{H^2(\mathbf{X})\} = \psi(\boldsymbol{\theta})$, and that $f_1(\mathbf{x}) = f(\mathbf{x};\, \boldsymbol{\theta}_1)$ and $f_2(\mathbf{x}) = f(\mathbf{x};\, \boldsymbol{\theta}_2)$. Then, with $w(\mathbf{X}) = f(\mathbf{X};\, \boldsymbol{\theta}_2)/f(\mathbf{X};\, \boldsymbol{\theta}_1)$, the variance of the weighted estimator $w(\mathbf{X})H(\mathbf{X})$ is given by

$$\mathrm{var}(w(\mathbf{X})H(\mathbf{X})) = E_{\boldsymbol{\theta}_1}\{(w(\mathbf{X})H(\mathbf{X}))^2\} - E^2_{\boldsymbol{\theta}_1}(w(\mathbf{X})H(\mathbf{X})).$$

Now, $E_{\boldsymbol{\theta}_1}(w(\mathbf{X})H(\mathbf{X})) = E_{\boldsymbol{\theta}_2}(H(\mathbf{X})) = \phi(\boldsymbol{\theta}_2)$, and

$$E_{\boldsymbol{\theta}_1}\{(w(\mathbf{X})H(\mathbf{X}))^2\}$$
$$= \int_{R_2} f^2(\mathbf{x};\, \boldsymbol{\theta}_2) H^2(\mathbf{x}) f(\mathbf{x};\, \boldsymbol{\theta}_1)/f^2(\mathbf{x};\, \boldsymbol{\theta}_1)\, d\mathbf{x}$$
$$= \int_{R_2} w(\mathbf{x}) H^2(\mathbf{x}) f(\mathbf{x};\, \boldsymbol{\theta}_2)\, d\mathbf{x}$$
$$= E_{\boldsymbol{\theta}_2}(w(\mathbf{X})H^2(\mathbf{X})),$$

since $w(\mathbf{x})$ is 0 outside of R_2.

For many of the common densities, $E_{\boldsymbol{\theta}_2}\{w(\mathbf{X})H^2(\mathbf{X})\}$ is expressible as a function of $\psi(\boldsymbol{\theta}^*)$ for some $\boldsymbol{\theta}^*$ in the parameter space of $\boldsymbol{\theta}$. Appendix A contains the expected value of $w(\mathbf{X})H^2(\mathbf{X})$ under f_2. These values may be used to investigate the efficiency of the procedure. For example, if f_1 and f_2 are both members of the beta family with parameters a and b, and if $H(x) = x$, then $\phi(a, b) = a/(a + b)$ and $\psi(a, b) = a(a + 1)/(a + b)(a + b + 1)$. From Appendix A the variance of the weighting scheme $w(\mathbf{X})H(\mathbf{X})$ is

$$\frac{\beta(a_1, b_1)\beta(2a_2 - a_1, 2b_2 - b_1)\psi(2a_2 - a_1, 2b_2 - b_1)}{\beta(a_2, b_2)^2} - (a_2/(a_2 + b_2))^2.$$

In particular, if f_1 is uniform, $a_1 = 1$, $b_1 = 1$, and $b_2 = 1$ (a series of these densities for $1 \le a \le 3$ is

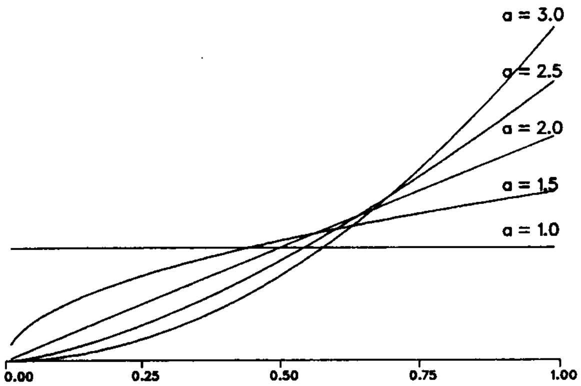

Figure 2. Example Densities.

given in Fig. 2), then the variance becomes

$$\mathrm{var}_{f_2}(w(X)H(X)) = a_2^4/(a_2 + 1)^2(2a_2 + 1).$$

Since the variance of $H(X)$ under f_2 is $a_2/(a_2 + 1)^2(a_2 + 2)$, the efficiency of the method as measured by the ratio of these variances is $(2a_2 + 1)/a_2^3(a_2 + 2)$.

By similar methods the efficiency of generating $H(x) = 1 - x$ under the density f_2 by weighting the observations from f_1 can be computed. A plot of the efficiency of the weighting method as a function of a for these two functions when $f_2(x) = ax^{(a-1)}$ and f_1 is uniform is given in Figure 3. The efficiency of weighting observations from the function $H(x) = x$ decreases rapidly as a increases. On the other hand, the efficiency for the function $H(x) = 1 - x$ not only increases over a portion of the range $1 \le a \le 3$, but is actually greater than one over the entire range. This is not surprising or unusual. The weighting scheme given here is similar to importance sampling in which one chooses the densities so that the resulting estimators have smaller variances than would have been attained by direct simulation. In importance sampling the experimenter is free to choose the density f_1 to minimize the variance. Here both f_1 and f_2 are determined by the densities of interest.

As illustration, 50 uniform 0–1 observations were generated and the mean values of both $H(X) = X$ and $H(X) = 1 - X$ were estimated for $f_2(x) = ax^{a-1}$. The estimated means as a function of a are shown as dots in Figure 4. The true means are denoted by the solid lines. For the function $H(x) = x$ in Figure 4(a), the inefficiency of the procedure forces what is a slight bias in the mean of the generated sample at $a = 1$ to become accentuated as a increases to 3. On the other hand, for $H(x) = 1 - x$ the bias at $a = 1$ has almost disappeared when a reaches 3. It is interesting to note that, even with the bias induced by sampling variability of the original sample, the estimated mean values follow the same general shape as the true means as a increases to 3. This property probably holds in general. Successive estimates all have the correct expectations that will force the correct shape. Systematic deviations from the true expected values are caused by the lack of independence of the estimators.

3. THE REJECTION METHOD

The second method of changing the input distribution relies on a random selection of the existing pairs of variables $(\mathbf{x}_i, y_i)$. It is necessary that there exist a uniform bound M such that $f_2(\mathbf{x})/f_1(\mathbf{x}) \le M$ for all $\mathbf{x}$, and that the support R_2 of f_2 be contained in the support R_1 of f_1. Let the random variable V given $\mathbf{X}_i = \mathbf{x}_i$ be uniform between 0 and $Mf_1(\mathbf{x}_i)$. The value $\mathbf{X}_i = \mathbf{x}_i$ and the corresponding $Y_i = y_i$ are retained as a sample from density $f_2(\mathbf{x})$ if the realization v of V is less than $f_2(\mathbf{x}_i)$.

To see why the rejection method produces the desired sampling distribution, consider the cdf of an $\mathbf{X}$ selected to remain in the sample.

$$\Pr\{\mathbf{X} \le \mathbf{z} \mid \mathbf{X} \text{ is retained}\} = \frac{\Pr\{\mathbf{X} \le \mathbf{z} \text{ and } \mathbf{X} \text{ is retained}\}}{\Pr\{\mathbf{X} \text{ is retained}\}} = \frac{\int_{\mathbf{u} \le \mathbf{z}} \Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} f_1(\mathbf{u})\, d\mathbf{u}}{\int_{R_1} \Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} f_1(\mathbf{u})\, d\mathbf{u}},$$

where $\mathbf{u} \le \mathbf{z}$ and $\mathbf{X} \le \mathbf{z}$ imply component by component inequality, and R_1 is the support for f_1. Now, $\Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} = f_2(\mathbf{u})/(Mf_1(\mathbf{u}))$, and the support R_2 of f_2 is contained in the support R_1 of f_1; otherwise M would not be finite. Hence

$$\int_{\mathbf{u} \le \mathbf{z}} \Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} f_1(\mathbf{u})\, d\mathbf{u} = \int_{\mathbf{u} \le \mathbf{z}} (f_2(\mathbf{u})/M)\, d\mathbf{u} = F_2(\mathbf{z})/M$$

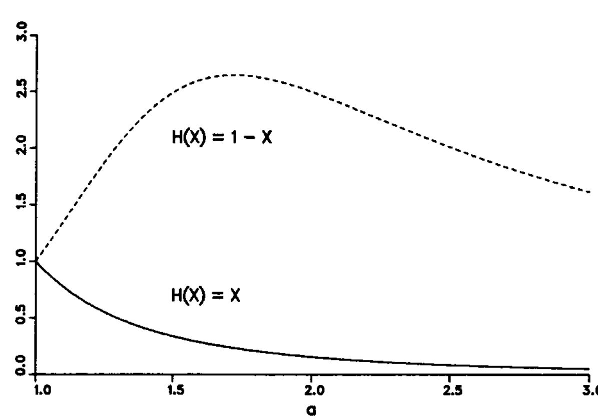

Figure 3. Efficiency of the Weighting Method.

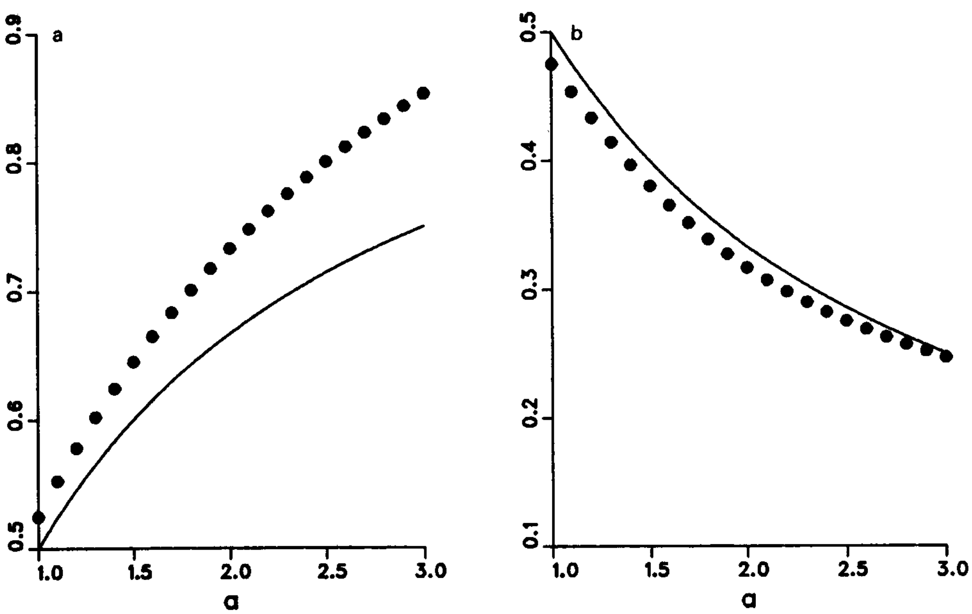

Figure 4. Simulated Mean Values: (a) H (X) = X; (b) H (X) = 1 − X.

and

$$\int_{R_1} \Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} f_2(\mathbf{u})\, d\mathbf{u}$$

$$= \int_{R_2} (f_2(\mathbf{u})/M)\, d\mathbf{u} = 1/M.$$

Therefore, $\Pr\{\mathbf{X} \le \mathbf{z} \mid \mathbf{X} \text{ is retained}\} = F_2(\mathbf{z})$, the distribution function of $\mathbf{X}$ under f_2.

This procedure for selecting a subset of the original sample is no more than the rejection method for random variate generation. In this application, however, the analyst cannot choose just any f_1 to generate efficiently samples from f_2, because f_1 is of interest of its own. Moreover, it is the unknown distribution of $H(\mathbf{X})$ that we are concerned with, not, as in the usual rejection methods of random variate generation, the distribution of $\mathbf{X}$.

The efficiency of this rejection method can be measured by the probability that a random $\mathbf{x}$ from f_1 is accepted as a random variate from f_2. This probability is given by the reciprocal of the bound M, since

$$\Pr\{\text{a random } \mathbf{X} \text{ is selected}\}$$
$$= \int_{R_1} \Pr\{V \le f_2(\mathbf{u}) \mid \mathbf{u}\} f_1(\mathbf{u})\, d\mathbf{u}$$
$$= 1/M.$$

As with the weighting method, the rejection method can be highly inefficient. Density functions that differ greatly will require large values of M. For some pairs of densities, $f_2(\mathbf{x})/f_1(\mathbf{x})$ may be unbounded.

Appendix B contains the bounds M for most of the densities commonly used in simulation studies. For f_1 uniform 0–1 and $f_2 = ax^{a-1}$, M from Appendix B is given by

$$M = \Gamma(a + 1)\Gamma(1)\Gamma(1)/\Gamma(2)\Gamma(a)\Gamma(1) = a.$$

Hence the efficiency as measured by the proportion of retained samples is $1/a$. Unless the original sample size is large, one would not use the rejection technique with these densities for values of a much in excess of 2, because the expected sample size is halved at that point.

As with the example given in Section 3, 50 samples were generated from a uniform 0–1. Empirical cdf's for the retained samples for the two functions $H(x) = x$ and $H(x) = 1 - x$, with $a = 1.5$ and $a = 2$, are given in Figure 5. The empirical distribution functions are shown as step functions with the true cdf superimposed. Each of the four empirical cdf's in this figure follows the general shape of the true cdf and is within the sampling variability expected for such estimators.

It is interesting to contrast the retained samples for the two functions. For $H(x) = x$, more samples whose realizations are greater than .5 are retained. From Figure 2 the densities with $a = 1.5$ and $a = 2$ are concentrated in the range from .5 to 1. Hence more large than small realizations are retained. For the function $H(x) = 1 - x$, more mass is to the left of .5 and more small realizations are retained.

Although losses in efficiency are guaranteed with

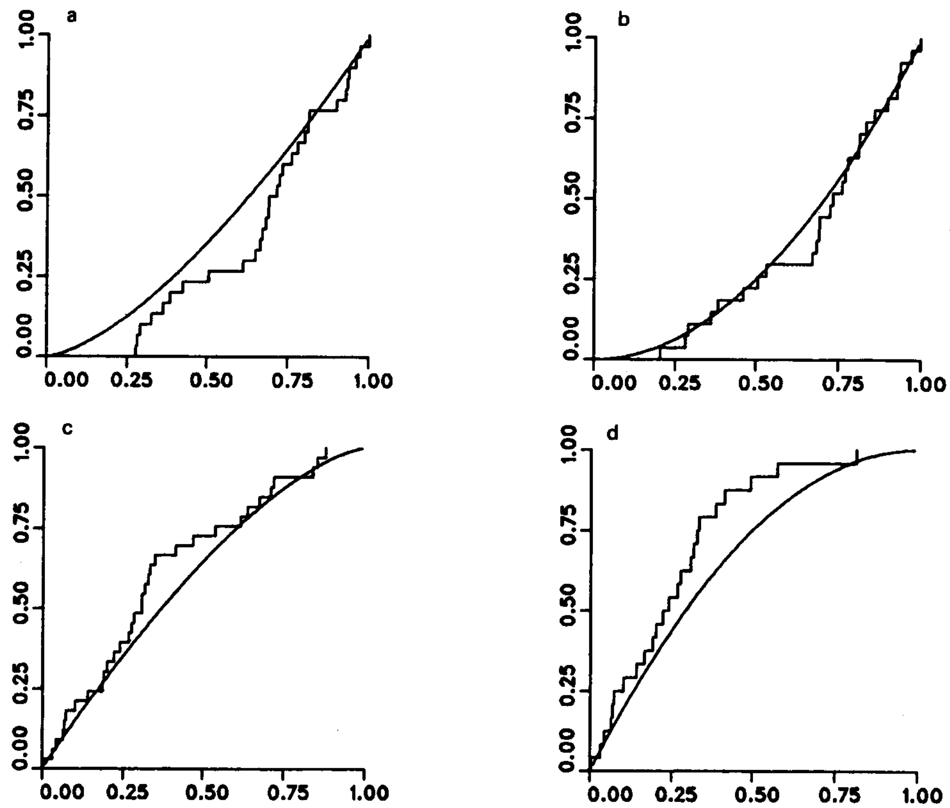

Figure 5. Empirical cdf's: (a) $H(X) = X$, $a = 1.5$; (b) $H(X) = X$, $a = 2.0$; (c) $H(X) = 1 - X$, $a = 1.5$; (d) $H(X) = 1 - X$, $a = 2.0$.

this method, the efficiency can remain high for some rather large changes in the shape of the densities. (This is the case in the example shown here in which a uniform is transformed to a triangular density.) Moreover, a low efficiency of the method may be a minor inconvenience when compared with rerunning the computer code.

4. EXAMPLE

The Safety Code Development Group (1985) in the Energy Division of Los Alamos National Laboratory wrote TRAC-PF1/MOD1, a transient reactor analysis code (TRAC) that provides predictions of postulated accidents in pressurized water reactors. The

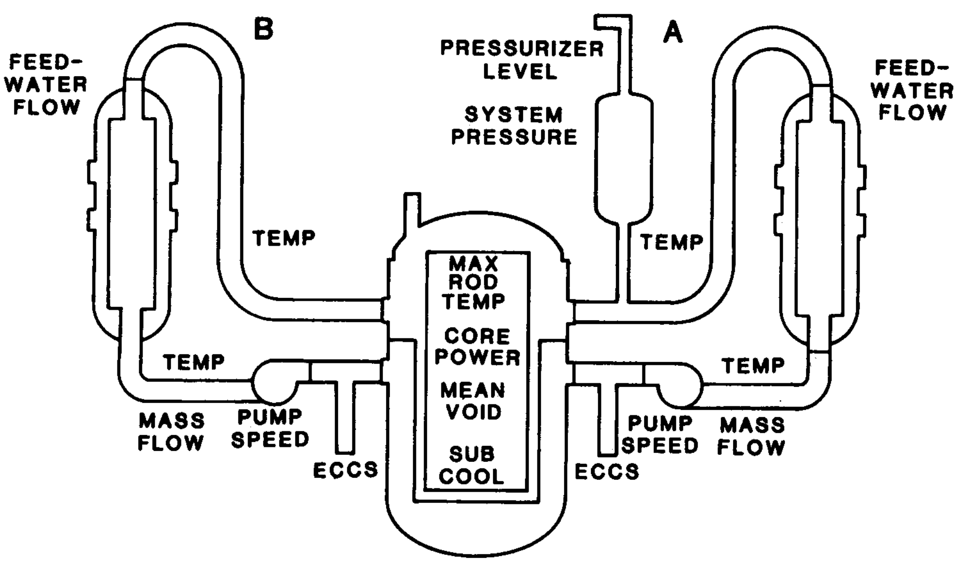

Figure 6. Simulated Reactor System Display From TRAC.

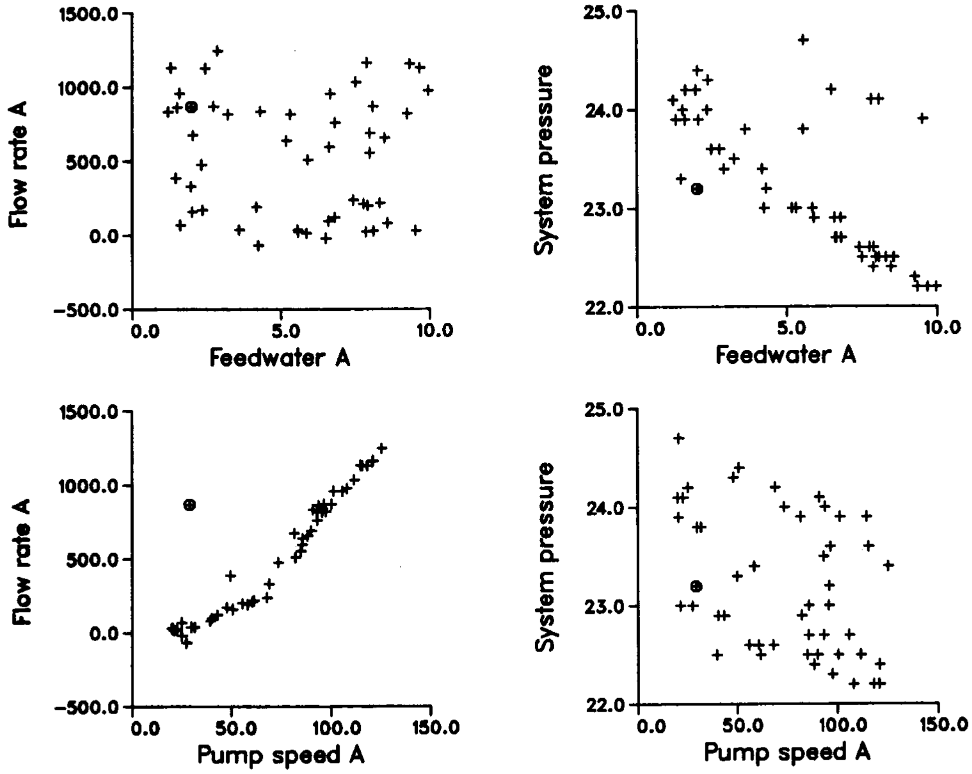

Figure 7. Input and Output Values From TRAC Simulation.

code is interactive in nature, producing computer terminal displays that allow reactor-safety personnel to monitor the behavior of the simulated reactor. A schematic drawing of such a display is shown in Figure 6. Included in the figure are variable names, such as mass flow rate and system pressure, that the operator is allowed to monitor. Also included are variables, such as pump speed and feedwater flow rate, that the operator may change.

For the purpose of an example, a simulation was performed using TRAC configured as the Three Mile Island Unit 2 reactor was on the day of its accident. Starting with this basic reactor configuration, 50 simulation runs were performed in which the pump speeds on the A side were drawn from a uniform density between 20 radians per second (rad/s) and 125.7 rad/s. Simultaneously, input values for feedwater flow rate on the A side were drawn from a uniform density with support from 1 kilogram per second (kg/s) to 10 kg/s. The changes in pump speed and flow rate were made after 2,600 (real time) seconds, and the code was then allowed to run an additional 1,000 (real time) seconds. Each run used approximately 5 minutes of CRAY-1 computer time.

At the conclusion of the 3,600 (real time) seconds the monitored variables were recorded. For this example, system pressure in M Pascal and mass flow rate on the A side in kg/s were typical of the monitored variables and are the ones presented here. Figure 7 shows the relationships between these output variables and the two input variables. Negative flow rates indicate backward flow through the system. The circled point is a case in which the computer code developed numerical problems and failed to run for the entire 3,600 seconds. This point was deleted from the analysis.

The density functions of interest in this study are the beta densities given in Figure 2 with the same support as the underlying uniforms given previously. We present the results obtained from changing the density of both input variables simultaneously from uniforms (f_1) to scaled densities of the form $f_2 = ax^{a-1}$. In actual studies one would not only change the densities simultaneously but would also make distributional changes to the input variables one at a time. In this way the total impact that each density exerts on the output variables can be assessed.

The results of this study are given in Figure 8. Using the weighting method the expected value of the A-side flow rate as a function of a is given in Figure 8(a). This expectation is increasing almost as a linear function of a. The empirical cdf's of the flow rate with $a = 1.5$ and $a = 2$ derived from the rejection method are given in Figure 8(b). As one would

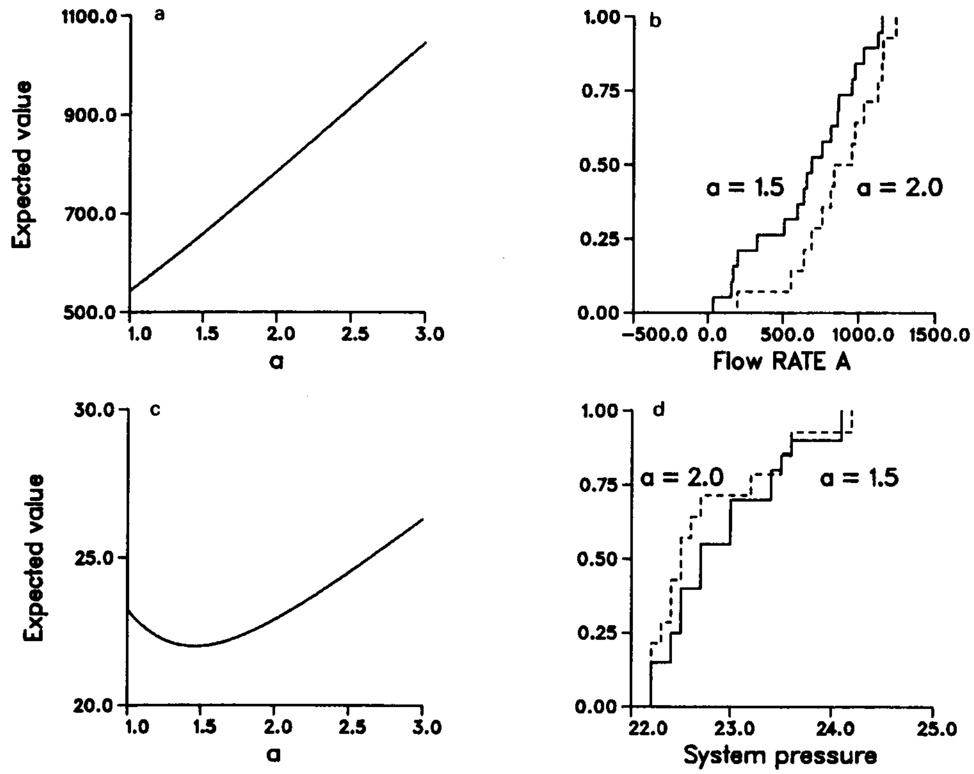

Figure 8. Estimation From TRAC Simulation: (a) Flow Rate A; (b) Empirical cdf; (c) System Pressure; (d) Empirical cdf.

expect from the increasing means in Figure 8(a), the cdf for $a = 2$ lies to the right of the flow rate cdf for $a = 1.5$.

The behavior of system pressure as a function of a is more interesting than that of flow rate. The estimated expected values of system pressure as a function of a obtained by the weighting method are given in Figure 8(c). This function is quadratic in form with a minimum close to $a = 1.5$. The empirical cdf's of system pressure as obtained from the rejection method for $a = 1.5$ and $a = 2$ are given in Figure 8(d). These cdfs are somewhat a surprise, as one expects the cdf at $a = 1.5$ to be to the left of the cdf when $a = 2$. There is some indication from Figure 8(d) that these distributions cross, and the distribution of system pressure when $a = 2$ has a long right tail compared to that when $a = 1.5$. Further simulations at $a = 2$ may be necessary to investigate this phenomenon completely.

The preceding simulation exercise shows the benefit of using the two methods presented here. It is impractical to study the distribution of the output variables at many alternative distributions of the input variables when each simulation study takes hours of computer time on even the fastest computers. The methods given here allow the experimenter to assess the practical implications of changing the input distributions and point out areas where further simulation may be necessary.

5. ACKNOWLEDGMENT

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APPENDIX A: $E_{\theta_2}\{wH^2(\mathbf{X})\}$ FOR SELECTED DENSITIES

Normal $N(\mu, \sigma^2)$

$$\exp\left[\frac{(\mu_1 - \mu_2)^2}{2\sigma_1^2 - \sigma_2^2}\right] \frac{\sigma_1^2}{[\sigma_2^2(2\sigma_1^2 - \sigma_2^2)]^{1/2}}\, \psi(a_1, a_2),$$

where $a_1 = (2\sigma_1^2\mu_2 - \sigma_2^2\mu_1)/(2\sigma_1^2 - \sigma_2^2)$ and $a_2 = \sigma_1^2\sigma_2^2/(2\sigma_1^2 - \sigma_2^2)$. Note that $E\{H^2\} = \psi(\mu, \sigma^2)$. The expectation does not exist in this form for $\sigma_2^2 > 2\sigma_1^2$.

Beta $x^{a-1}(1-x)^{b-1}/\beta(a, b)$

$$\frac{\beta(a_1, b_1)\beta(2a_2 - a_1, 2b_2 - b_1)\psi(2a_2 - a_1, 2b_2 - b_1)}{\beta^2(a_2, b_2)}.$$

Note that $E\{H^2\} = \psi(a, b)$. The expectation does not exist in this form for $a_1 > 2a_2$ or $b_1 > 2b_2$.

Gamma $x^{\alpha-1}e^{-x/\beta}/\Gamma(\alpha)\beta^{\alpha}$

$$\psi(z_1, z_2)\Gamma(\alpha_1)\beta_1^{\alpha_1}\Gamma(z_1)z_2^{(2\alpha_2-\alpha_1)}/\Gamma^2(\alpha_2)\beta_2^{2\alpha_2},$$

where $z_1 = 2\alpha_2 - \alpha_1$ and $z_2 = \beta_1\beta_2/(2\beta_1 - \beta_2)$. Note that $E\{H^2\} = \psi(\alpha, \beta)$. The expectation does not exist in this form for $\alpha_1 > 2\alpha_2$ or $\beta_2 > 2\beta_1$.

Poisson $\lambda^x\varepsilon^{-\lambda}/x!$

$$\exp\{(\lambda_2 - \lambda_1)^2/\lambda_1\}\psi(\lambda_2^2/\lambda_1).$$

Note that $E\{H^2\} = \psi(\lambda)$.

Binomial $B(N, p)$

$$\left[\frac{p_2^2 - 2p_1p_2 + p_1}{(1-p_1)p_1}\right]^N \psi\left(\frac{p_2^2(1-p_1)}{p_2^2 - 2p_1p_2 + p_1}\right).$$

Note that $E\{H^2\} = \psi(p)$ and $N_1 = N_2 = N$.

Exponential Family $C(\boldsymbol{\theta})B(x)\exp(\Sigma\theta_jT_j(x))$

$$C^2(\boldsymbol{\theta}_2)/[C(\boldsymbol{\theta}_1)C(2\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)]\psi(2\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1).$$

Note that $E\{H^2\} = \psi(\boldsymbol{\theta})$. This form exists only when $C(2\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)$ is defined.

APPENDIX B: THE BOUND M FOR SELECTED DENSITIES

Normal $N(\mu, \sigma^2)$

For $\sigma_1^2 > \sigma_2^2$,

$$M = (\sigma_1/\sigma_2)\exp(.5(\mu_2 - \mu_1)^2/(\sigma_1^2 - \sigma_2^2)).$$

Beta $x^{a-1}(1-x)^{b-1}/\beta(a, b)$

For (a) $a_2 > a_1$ and $b_2 > b_1$ or (b) $a_2 < a_1$, $b_2 > b_1$, and $(a_1 - a_2) > (b_2 - b_1)$,

$$M = CX_0^{(a_2-a_1)}(1 - X_0)^{(b_2-b_1)},$$

where $X_0 = (a_2 - a_1)/(a_2 - a_1 + b_2 - b_1)$ and $C = \beta(a_1, b_1)/\beta(a_2, b_2)$.

For (c) $a_1 = a_2$ and $b_2 > b_1$ or (d) $a_2 > a_1$ and $b_1 = b_2$,

$$M = \beta(a_1, b_1)/\beta(a_2, b_2).$$

Gamma $x^{\alpha-1}e^{-x/\beta}/\Gamma(\alpha)\beta^{\alpha}$

For $\alpha_2 > \alpha_1$ and $\beta_1 > \beta_2$,

$$M = CX_0^{(\alpha_2-\alpha_1)}\exp\{(1/\beta_1 - 1/\beta_2)X_0\},$$

where $X_0 = \beta_1\beta_2(\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)$ and $C = \Gamma(\alpha_2)\beta_1^{\alpha_1}/\Gamma(\alpha_1)\beta_2^{\alpha_2}$.

For $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 > \beta_2$,

$$M = (\beta_1/\beta_2)^{\alpha}.$$

Poisson $\lambda^x e^{-\lambda}/x!$

For $\lambda_1 > \lambda_2$,

$$M = \exp(\lambda_1 - \lambda_2).$$

Binomial $B(N, p)$

For $p_1 > p_2$ and $N_1 > N_2$,

$$M = (1 - p_2)^{N_2}/(1 - p_1)^{N_1}.$$

For $p_1 = p_2 = p$ and $N_1 > N_2$,

$$M = (1 - p)^{N_2 - N_1}.$$

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